Math 564: Advance Analysis 1
Lecture 10

We proved that continuous functions are Borel (hence measurable), which is a special case of the following statement, whose proof is the same as for continuous functions.

Prop. Let $(X, \mathcal{I})$, $(Y, \mathcal{J})$ be measurable spaces and let $\mathcal{J}_0 \subseteq \mathcal{J}$ be generating $\mathcal{J}$ as a σ-algebra. If a function $f: X \to Y$ is such that $f^{-1}(\mathcal{J}_0) \subseteq \mathcal{I}$ (i.e. $f^{-1}(J_0) \in \mathcal{I}$ for each $J_0 \in \mathcal{J}_0$), then f is $(\mathcal{I}, \mathcal{J})$-measurable.

Theorem. Let (X, μ) be a topological space with Borel measure μ and let Y be a 2^{nd} ctbl top. space (e.g. $\mathbb{R}$).

(a) Every μ-measurable $f: X \to Y$ is almost Borel, i.e. $\exists$ Borel conull $X' \subseteq X$ s.t. $f|_{X'}$ is Borel. In particular, $f \approx^{\mu}$ a Borel function g, i.e. $f = g$ a.e.

(b) Suppose that (X, μ) is strongly regular (e.g. X is metric and ctbl union of finite measure open sets, like $\mathbb{R}^d$ with Lebesgue measure).

Luzin's Theorem

Every μ-meas. $f: X \to Y$ is ε-almost continuous, i.e. $\exists$, say, closed $X' \subseteq X$ with $\mu(X \setminus X') \le \varepsilon$ s.t. $f|_{X'}$ is continuous.

Proof. Let $f: X \to Y$ be μ-meas. and let $\{V_n\}$ be a ctbl (open) basis of Y.

(a) By the prop. above, it's enough to make each $f^{-1}(V_n)$ Borel. So for each n, take $f^{-1}(V_n) \approx^{\mu} B_n$, where B_n is Borel. Then $Z := \bigcup_n (f^{-1}(V_n) \triangle B_n)$ is null, so $\exists$ null Borel set $\hat{Z} \supseteq Z$. Let $X' := X \setminus \hat{Z}$. Then $f|_{X'}: X' \to Y$ is Borel because $(f|_{X'})^{-1}(V_n) = f^{-1}(V_n) \cap X' = B_n \cap X'$ is Borel. Define $g: X \to Y$ by setting $g|_{X'} := f|_{X'}$ and $g|_{\hat{Z}} :=$ constant $y_0 \in Y$. This is Borel and $= f$ on the small set X'.

(b) Here we want to make $f^{-1}(V_n)$ open and we do:
Let $f^{-1}(V_n) \approx^{\mu}_{\varepsilon/2^{n+2}} U_n$ open (by strong regularity). Then

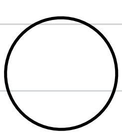
$Z := \bigcup_n \left(f^{-1}(V_n) \triangle U_n\right)$ has measure $\leq \varepsilon/2$ so $\exists$ open $\hat{Z} \supseteq Z$ of measure $\leq \varepsilon$. Then $X' := X \setminus \hat{Z}$ and $f|_{X'}$ is continuous because $(f|_{X'})^{-1}(V_n) = f^{-1}(V_n) \cap X' = U_n \cap X'$ is open inside X'. $\square$

Push-forward measures. Let $(X, \mathcal{I}), (Y, \mathcal{J})$ be measurable spaces, let $f : X \to Y$ be an $(\mathcal{I}, \mathcal{J})$-measurable function.
For each measure μ on $\mathcal{I}$, we define its push-forward through f to a measure $f_*\mu$ on $\mathcal{J}$ by setting, for each $J \in \mathcal{J}$,

$$f_*\mu(J) := \mu(f^{-1}(J)).$$

Because f^{-1} commutes with ctbl disjoint unions, this is indeed a measure, and such that $f_*\mu(Y) = \mu(X)$.

Examples. (a) Let S^1 denote circle (inside $\mathbb{C}$ or $\cong \mathbb{R}/\mathbb{Z}$), so it is a compact Hausdorff group, hence a unique Haar prob. meas.



We can explicitly construct this Haar measure by setting it equal to $\frac{1}{2\pi}$ arc-length on arcs and taking Carathéodory extension. But, we can also obtain this measure as the push-forward through $\exp : [0,1) \to S^1$ by $x \mapsto e^{2\pi i x}$ of Lebesgue measure on $[0,1)$.

(b) Let $(X, \mathcal{I}), (Y, \mathcal{J})$ be measurable spaces. For any measure μ on $(X \times Y, \mathcal{I} \otimes \mathcal{J})$, where $\mathcal{I} \otimes \mathcal{J} :=$ the σ-alg. gen. by sets $I \times J$ where $I \in \mathcal{I}$ and $J \in \mathcal{J}$, has push-forwards μ_X on $\mathcal{I}$ and μ_Y on $\mathcal{J}$

through the projection maps $\mathrm{proj}_X : X \times Y \to X$ and $\mathrm{proj}_Y : X \times Y \to Y$. μ_X and μ_Y are called the marginals of μ. Conversely this μ is call a joining of μ_X and μ_Y.

(c) Let $\mathrm{Graphs}(\mathbb{N}) :=$ the set of all undirected graphs on $\mathbb{N} \cong \mathcal{P}([\mathbb{N}]^2) \cong 2^{[\mathbb{N}]^2}$, where $[\mathbb{N}]^2 :=$ the set of all 2-element subsets of $\mathbb{N} \cong \{(n,m) \in \mathbb{N}^2 : n<m\}$. A random graph on $\mathbb{N}$ is a prob. meas. on $\mathrm{Graphs}(\mathbb{N}) \cong 2^{[\mathbb{N}]^2}$. In probability, we think of these measures as push-forward through a μ-measurable map

$$G : \underset{\text{seed/configuration}}{\Omega} \to \mathrm{Graphs}(\mathbb{N}), \quad \omega \mapsto G(\omega)$$

where (Ω, μ) is a prob. space.

By a random graph they mean $G(\omega)$ of some "random" $\omega \in \Omega$. And by the law of G they mean the push-forward $G_* \mu$. The random graph (aka the Rado graph, aka Erdős–Rényi graph) is just the $\mathrm{Bernoulli}(\frac{1}{2})$ measure on $2^{[\mathbb{N}]^2}$. It turns out that a μ-null set of these graphs are isomorphic to each other so there is one graph on $\mathbb{N}$ up to isomorphism that is known as the random graph.

Borel/measure isomorphism theorems.

Def. (a) For top. spaces X, Y, a Borel isomorphism is a bij $f: X \to Y$ s.t. f and f^{-1} are Borel. (If X, Y are Polish, then f being Borel implies that f^{-1} is Borel.)

(b) For measure spaces (X, μ) and (Y, ν), a measure isomorphism is a function $f : X \to Y$ s.t. $\exists$ μ-conull $X' \subseteq X$, ν-conull $Y' \subseteq Y$ where $f|_{X'} : X' \to Y'$ is a bijection, $f|_{X'}$ and $f^{-1}|_{Y'}$ are μ and ν measurable, respectively, and $f_* \mu = \nu$ (so $f^{-1}_* \nu = \mu$).